This week

1. Section 5.1: area estimating with finite sums
2. Section 5.2: limits of finite sums
3. Section 5.3: the definite integral
4. Section 5.4: the fundamental theorem of calculus
The Σ-notation 1.1

We can write sums in a shorter way using the Σ-notation:

\[
\sum_{k=M}^{N} a_k = a_M + a_{M+1} + a_{M+2} + \cdots + a_{N-1} + a_N
\]

- Σ is the Greek letter “S” (pronounced as 'sigma'), which refers to “Sum”.
- \(k\) is the index.
- The index starts at \(M\) and ends at \(N\).
- \(a_k\) is the \(k\)-th term of the sum, and is a formula containing \(k\).
- If \(N < M\) then the sum is equal to 0 by convention.
- The index is a dummy:

\[
\sum_{k=3}^{6} a_k = \sum_{p=3}^{6} a_p = a_3 + a_4 + a_5 + a_6
\]

The Σ-notation 1.2

\[
\sum_{k=1}^{12} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2
\]
\[
= 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 + 144
\]
\[
= 650.
\]
Examples

1. \[\sum_{k=1}^{4} (-1)^k k = (-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4\]

\[= -1 + 2 - 3 + 4 = 2.\]

2. \[\sum_{k=1}^{2} \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1}\]

\[= \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.\]

Arithmetic series

Theorem

The sum of the first \(n\) positive integers is equal to \(\frac{n(n+1)}{2}\).

- with \(\Sigma\)-notation:
  \[1 + 2 + \cdots + (n-1) + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.\]

- Write out the terms in the sum twice:
  \[
  \begin{array}{cccccc}
  1 & + & 2 & + & \cdots & + (n-1) + n \\
  n & + & (n-1) & + & \cdots & + 2 + 1 \\
  \hline
  (n+1) & + & (n+1) & + & \cdots & + (n+1) + (n+1)
  \end{array}
  \]

- Adding the columns gives \(n\) terms, all equal to \(n + 1\), so
  \[2 \sum_{k=1}^{n} k = n(n+1).\]
Rules

- **Sum- and difference rule:**
  \[ \sum_{k=M}^{N} (a_k + b_k) = \sum_{k=M}^{N} a_k + \sum_{k=M}^{N} b_k, \text{ and } \sum_{k=M}^{N} (a_k - b_k) = \sum_{k=M}^{N} a_k - \sum_{k=M}^{N} b_k. \]

- **Constant multiple rule:**
  \[ \sum_{k=M}^{N} c a_k = c \sum_{k=M}^{N} a_k. \]

- **Constant value rule:**
  \[ \sum_{k=M}^{N} c = (N - M + 1)c. \]

- **Splitting rule:**
  \[ \sum_{k=M}^{N} a_k = \sum_{k=M}^{P} a_k + \sum_{k=P+1}^{N} a_k. \]

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Example

- **Define** \( S_n \) **as the sum of the first** \( n \) **odd integers:**
  \[ S_n = 1 + 3 + \cdots + (2n - 1) \]

- **Notice that**
  \[ S_n + (2 + 4 + \cdots + 2n) = 1 + 2 + 3 + \cdots + (2n - 1) + 2n = \frac{2n(2n + 1)}{2} = n(2n + 1) = 2n^2 + n \]

- **Furthermore**
  \[ 2 + 4 + \cdots + 2n = \sum_{k=1}^{n} 2k = 2 \sum_{k=1}^{n} k = 2 \cdot \frac{n(n+1)}{2} = n(n + 1) = n^2 + n. \]

- **Therefore**
  \[ S_n = (2n^2 + n) - (n^2 + n) = n^2. \]
Example

The sum of the first $n$ odd integers is equal to $n^2$:

\[
\begin{array}{cccccc}
1 & 3 & 5 & 7 & 9 & 11 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{array}
\]

1.8 Shifting the index

Note that

\[1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{k=1}^{5} k^2,\]

but also

\[1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{m=0}^{4} (m + 1)^2,\]

and even

\[1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{h=3}^{7} (h - 2)^2.\]

Since the index is a dummy, you could even write

\[\sum_{k=1}^{5} k^2 = \sum_{k=0}^{4} (k + 1)^2 = \sum_{k=3}^{7} (k - 2)^2.\]
Assignment: IMM1 - Tutorial 6.1

Partitions

Definition

A partition of the interval \([a, b]\) in \(n\) subintervals is a sequence \(x_0, x_1, \ldots, x_n\) constructed as follows:

(i) \(\Delta x = \frac{b - a}{n}\)

(ii) \(x_k = a + k\Delta x\) \((k = 0, 1, \ldots, n)\)

- Note that \(x_0 = a, \quad x_n = b, \quad x_k - x_{k-1} = \Delta x.\)
- The number \(\Delta x\) is called the mesh of the partition.
Riemann sums

Partition \([a, b]\) in \(n\) subintervals.

**Definition**

We define the \(n\)-th Riemann sum of \(f\) over \([a, b]\) as

\[
\sum_{k=1}^{n} f(x_k) \cdot \Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n}
\]

Computing displacement from velocity

Consider a moving object and assume that we know its velocity as a function of time \(v(t)\).

Can we compute the displacement using the function \(v(t)\)?

- If \(v(t) = v_0\) is constant, then the displacement is equal to the product of \(v_0\) and the elapsed time.
- If \(v(t)\) is not constant, then we approximate the displacement with a Riemann sum.
Computing displacement from velocity

Assume that we know the velocity $v(t)$ for $a \leq t \leq b$, then we can find an estimate for the displacement while $t$ elapses from $a$ to $b$.

1. Partition the interval $[a, b]$ in $n$ subintervals with mesh $\Delta t = \frac{b - a}{n}$ and intermediate points $t_k = a + k\Delta t$.

2. While $t$ runs from $a = t_0$ to $t_1$, the displacement is approximately $v(t_1)\Delta t$;
   - while $t$ runs from $t_1$ to $t_2$, the displacement is approximately $v(t_2)\Delta t$;
   etcetera.

3. The total displacement is approximately

$$\sum_{k=1}^{n} v(t_k)\Delta t.$$
Approximate the area of the triangle with vertices \((0, 0), (1, 0)\) and \((1, 1)\) with a Riemann sum.

- Define the partition \(x_k = k\Delta x = \frac{k}{n}\) with \(\Delta x = \frac{1}{n}\).
- The Riemann sum of \(f(x) = x\) is
  \[
  \sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n}.
  \]

If we let \(n\) approach infinity then
  \[
  \lim_{n \to \infty} \sum_{k=1}^{n} x_k \Delta x = \frac{1}{2}.
  \]

Evaluate the Riemann sum:
  \[
  \sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{n^2} \frac{n(n + 1)}{2} = \frac{1}{2} \left(1 + \frac{1}{n}\right) = \frac{1}{2} + \frac{1}{2n}.
  \]

If we let \(n\) approach infinity then
  \[
  \lim_{n \to \infty} \sum_{k=1}^{n} x_k \Delta x = \frac{1}{2}.
  \]
For a positive function, a Riemann sum can be regarded as the approximation of the surface area of the region $R$ bounded by the graph of $f$, the $x$ axis, and the lines $x = a$ and $x = b$.

**Definition**

The definite integral of $f$ over the interval $[a, b]$ is defined as

$$
\int_a^b f(x) \, dx = \lim_{n \to \infty} \left( \sum_{k=1}^{n} f(x_k) \cdot \Delta x \right)
$$

A definite integral can be regarded as the area of the region $R$.

**Laws of integration**

- The variable in the integral is a **dummy**:

  $$
  \int_a^b f(x) \, dx = \int_a^b f(u) \, du
  $$

- Linearity:

  $$
  \int_a^b \alpha f(x) + \beta g(x) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx
  $$

- Additivity:

  $$
  \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
  $$

- Interchanging the upper and lower limit gives a minus sign:

  $$
  \int_a^b f(x) \, dx = - \int_b^a f(x) \, dx
  $$
If a function is negative on an interval, then the integral over that interval is negative.

The integral adds the areas of the positive part of $f$, but subtracts the areas of the negative parts:

$$\int_a^b f(x) \, dx = \text{Area}(A) - \text{Area}(B).$$

Notice that the Riemann sum of any partition is

$$\sum_{k=1}^n c \Delta x = n \cdot c \Delta x = c \frac{b - a}{n} = c(b - a).$$
Laws of integration

\[ \int_a^b x \, dx = \frac{1}{2} b^2 - \frac{1}{2} a^2 \]

Note that \( \frac{1}{2} b^2 - \frac{1}{2} a^2 = \frac{1}{2} (b + a)(b - a) = \text{Area}(R) \).
### Displacement and velocity

- Differentiate displacement to compute velocity:
  \[ v(t) = s'(t) \]

- The displacement can be computed from the velocity by integrating:
  \[
  s(t) = \lim_{n \to \infty} \sum_{k=1}^{n} v(t_k) \Delta t = \int_{0}^{t} v(\tau) \, d\tau
  \]

  The integral \( \int_{0}^{t} v(\tau) \, d\tau \) is a function \( s(t) \) whose derivative is \( v \).

### Charge and current

- Add the charges in all compartments:
  \[ Q(t) = \sum_{k=1}^{n} i(t_k) \Delta t. \]

- The total charge passing through \( A \) is
  \[
  Q(t) = \lim_{n \to \infty} \sum_{k=1}^{n} i(t_k) \Delta t = \int_{0}^{t} i(\tau) \, d\tau.
  \]

- Current is the rate of change of charge:
  \[ i(t) = Q'(t). \]
Antiderivatives

Definition

We call a function $F$ an antiderivative for $f$ if $F'(x) = f(x)$.

- Antiderivatives are not unique. If $F$ is an antiderivative for $f$, then so is $F(x) + C$ for any constant $C$:
  \[
  \frac{d}{dx}(F(x) + C) = F'(x) = f(x).
  \]

Theorem

Let $(x_0, y_0)$ be a point in the plane. Then there is a unique antiderivative $F$ of $f$ for which $F(x_0) = y_0$.

Example

- Let $f(x) = e^x + 1$, then $F(x) = e^x + x$ is an antiderivative of $f$.
- For arbitrary $C$ the function
  \[
  F_c(x) = e^x + x + C
  \]
  is also an antiderivative of $f$.
- There is only one antiderivative of $f$ for which $F(0) = 4$:
  \[
  F(x) = e^x + x + 3.
  \]
  The correct value for $C$ is found by solving the equation $F_c(0) = 4$:
  \[
  4 = F_c(0) = e^0 + 0 + C = 1 + C,
  \]
  hence $C = 3$. 

The inverse of differentiation

The Fundamental Theorem of Calculus

1. Define the function
   \[ F(x) = \int_a^x f(t) \, dt, \]
   then \( F \) is an antiderivative for \( f \), in other words: \( F'(x) = f(x) \).

2. If \( F \) is an antiderivative for \( f \) then
   \[ \int_a^b f(t) \, dt = F(b) - F(a). \]

Notation: \( F(b) - F(a) = \left[ F(x) \right]_a^b = \left. F(x) \right|_a^b \).

The function \( F(x) = \int_a^x f(t) \, dt \) also satisfies \( F(a) = 0 \), so \( F \) is the unique antiderivative of \( f \) for which \( F(a) = 0 \).
Integrals with $\sin$ and $\cos$

\[ \int_{a}^{b} f(x) \, dx = F(x) \bigg|_{a}^{b} = F(b) - F(a) \quad \text{where } F' = f. \]

\[ \int_{0}^{\pi/2} \sin(x) \, dx = -\cos(x) \bigg|_{0}^{\pi/2} = -\cos \left( \frac{\pi}{2} \right) - (-\cos 0) = 0 - (-1) = 1 \]

\[ \int_{\pi}^{2\pi} \sin(x) \, dx = -\cos(x) \bigg|_{\pi}^{2\pi} = -\cos(2\pi) - (-\cos(\pi)) = 1 - (-(-1)) = -2 \]

The antiderivative of $\cos(x)$

\[ \int_{0}^{x} \cos(t) \, dt = \sin(t) \bigg|_{t=0}^{x} = \sin(x). \]
The antiderivative of $\sin(x)$

$$\int_0^x \sin(t) \, dt = -\cos(t) \bigg|_{t=0}^x = -\cos(x) - (-1) = -\cos(x) + 1.$$

### Power functions

**Notice that for arbitrary real $\alpha$ we have**

$$\frac{d}{dx} \left(x^{\alpha+1}\right) = (\alpha + 1)x^\alpha.$$  

**Hence, if $\alpha \neq -1$:**

$$\frac{d}{dx} \left(\frac{1}{\alpha + 1} x^{\alpha+1}\right) = x^\alpha.$$  

The antiderivative of $x^\alpha$ is:  

$$\frac{1}{\alpha + 1} x^{\alpha+1} + C \quad \text{if } \alpha \neq -1.$$  

The antiderivative of $x^{-1} = \frac{1}{x}$ is:  

$$\ln |x| + C.$$  

See lecture 5
Polynomials

\[ \int_0^1 2x^3 - 2x + 1 \, dx = \int_0^1 2x^3 \, dx - \int_0^1 2x \, dx + \int_0^1 1 \, dx \]

\[ = \left[ \frac{1}{2}x^4 \right]_0^1 - \left[ x^2 \right]_0^1 + \left[ x \right]_0^1 \]

\[ = \left( \frac{1}{2} \cdot 1^4 - \frac{1}{2} \cdot 0^4 \right) - (1^2 - 0^2) + (1 - 0) \]

\[ = \frac{1}{2} - 1 + 1 = \frac{1}{2}. \]